UNIVERSAL FINITE GROUP EXTENSIONS AND A NON-SPLITTING THEOREM†

BY

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ABSTRACT

Let G and K be finite groups whose orders have a common prime divisor. Then there is a group K^* closely related to K for which there is a non-split extension of K^* by G.

In [8] the problem of splitting of group extensions was considered from the following point of view: given a group K and a suitable class $\mathfrak{X}$ of groups, under what circumstances do all extensions of K by $\mathfrak{X}$-groups split? In this note, we make some remarks about the dual question: given a group G and a class $\mathfrak{X}$ of groups, when do all extensions of $\mathfrak{X}$-groups by G split? We discuss only the case in which G is a finite group and $\mathfrak{X}$ a class of finite groups.

A first relevant fact is a result of W. Gaschütz [1]. If G is a finite group and p a prime divisor of the order $|G|$ of G then there is a finite group H with a normal elementary abelian p-subgroup $A \neq 1$ such that $H/A \cong G$ and $A \leqq \Phi(H)$, the Frattini subgroup of H; hence such that H does not split over A. Therefore, if $\mathfrak{X}$ is any class of finite groups which contains all elementary abelian p-groups for all primes p which divide the orders of $\mathfrak{X}$-groups, then a necessary (and of course, by the Schur-Zassenhaus theorem, sufficient) condition for all extensions of $\mathfrak{X}$-groups by G to split is that all $\mathfrak{X}$-groups have orders co-prime to $|G|$. We shall prove the following sharper non-splitting result:

THEOREM 1. *Let G and K be finite groups such that $(|G|,|K|) > 1$. Then there is a non-split extension of a group K^* by G, where K^* is a subgroup of*

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a finite direct product of copies of K, and K is an epimorphic image of K^*. In particular, for every prime p, the Sylow p-subgroups of K and K^* have the same class, derived length and exponent; and if K is soluble, K and K^* have the same derived length, nilpotent length and p-length for all primes p.

On the other hand, we cannot in general choose K^* in Theorem 1 to be a direct product of copies of K, in view of the following simple fact.

THEOREM 2. *The class of finite groups all extensions of which split is closed under the formation of finite direct products.*

This class certainly contains non-trivial groups since it contains for instance all complete finite groups; it also contains groups which are not complete: see [8].

In order to prove Theorem 1, we need a straightforward generalization of a fundamental result of Gaschütz [1]. Before stating this, we introduce some notation and terminology. We use P. Hall's convenient notion of closure operations on classes of groups: see [4, §1.3]. Thus a class $\mathfrak{X}$ of groups is said to be S-closed if every subgroup of an $\mathfrak{X}$-group is an $\mathfrak{X}$-group, D_0-closed if the direct product of any 2 $\mathfrak{X}$-groups is an $\mathfrak{X}$-group, and R_0-closed if any subdirect product of 2 $\mathfrak{X}$-groups is an $\mathfrak{X}$-group. If t is a positive integer, a group K is said to be a t-generator group if K has a generating set of elements with at most t members. We shall call a class $\mathfrak{X}$ of finite groups *bounded* if, for every positive integer t, there is a corresponding positive integer $X(t)$ such that all t-generator $\mathfrak{X}$-groups have orders $\leqq X(t)$.

The generalization of Gaschütz's result which we shall use is

THEOREM 3. *Let n be a positive integer, G a finite n-generator group and $\mathfrak{X}$ a bounded R_0-closed* class *of finite groups. Let $\mathscr{C}$ be the class of all group extensions $1 \to K \xrightarrow{\iota} H \to G \to 1$ (where ι denotes the inclusion map) such that K is an $\mathfrak{X}$-group and H an n-generator group. Then there is in $\mathscr{C}$ an extension $1 \to K^* \to H^* \to G \to 1$ which is universal for $\mathscr{C}$ in the following sense. For any extension $1 \to K \to H \to G \to 1$ in $\mathscr{C}$ there is an epimorphism $\chi\colon H^* \to H$ such that the diagram below is commutative.*

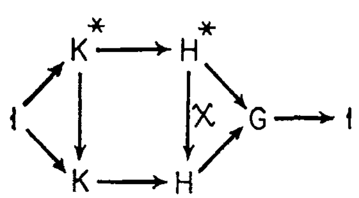

Fig. 1

In this connexion, see the remarks at the beginning of Chap. 9 of K. W. Gruenberg's book [3].

REMARK. In the diagram above, since χ is an epimorphism, χ maps K^* onto K. In order to prove Theorem 3 we first prove

LEMMA 1. *Let n be a positive integer and F a free group of rank n. Let G and H be n-generator groups for which there is an epimorphism $\zeta: H \to G$ such that* Ker ζ *is finite. Then, for any epimorphism $\theta: F \to G$, there is an epimorphism $\eta: F \to H$ such that*

$$\theta = \eta\zeta.$$

The appropriate diagram is

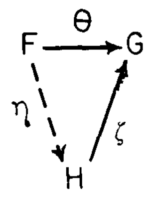

Fig. 2

PROOF. Let $\{x_1, \cdots, x_n\}$ be a set of free generators of F and let

$$x_j\theta = g_j \text{ for } j = 1, \cdots, n.$$

Then

$$G = \langle g_1, \cdots, g_n \rangle.$$

Since Kerζ is finite and H is an n-generator group, it follows from a result of Gaschütz ([2, Satz 1]) that there is a set $\{h_1, \cdots, h_n\}$ of generators of H such that

$$h_j\zeta = g_j \quad \text{for } j = 1, \cdots, n.$$

Now there is a (unique) homomorphism $\eta: F \to H$ such that

$$x_j\eta = h_j \quad \text{for } j = 1, \cdots, n.$$

Since $H = \langle h_1, \cdots, h_n \rangle$ and $F = \langle x_1, \cdots, x_n \rangle$, it follows that η is an epimorphism and

$$\eta\zeta = \theta.$$

PROOF OF THEOREM 3. Let F be a free group of rank n and let

$$1 \to R \to F \xrightarrow{\theta} G \to 1$$

be a presentation of G. By Schreier's theorem, R is finitely generated. Therefore, since $\mathfrak{X}$ is a bounded class, the quotient groups of R which are $\mathfrak{X}$-groups have bounded orders. We choose a normal subgroup T of R such that R/T is an $\mathfrak{X}$-group of maximal order. Then, since $\mathfrak{X}$ is R_0-closed, T is in fact the unique smallest normal subgroup of R with an $\mathfrak{X}$-group as quotient. Hence T is normal in F.

Let $\bar{\theta}$ be the homomorphism of F/T onto G induced by θ. Then

$$1 \to R/T \to F/T \xrightarrow{\bar{\theta}} G \to 1$$

is an extension in the class $\mathscr{C}$. We claim that it is universal for $\mathscr{C}$ in the sense defined.

Let

$$1 \to K \to H \xrightarrow{\zeta} G \to 1$$

be any extension in $\mathscr{C}$. Since G and H are n-generator groups and $\operatorname{Ker}\zeta = K$, which is finite, we can apply Lemma 1. This guarantees the existence of a presentation of H, say

$$1 \to S \to F \xrightarrow{\eta} H \to 1$$

such that

$$\theta = \eta\zeta.$$

Then

$$S = \operatorname{Ker}\eta \leqq \operatorname{Ker}\theta = R.$$

Moreover, η induces an isomorphism of F/S onto H in which R/S is mapped to $\operatorname{Ker}\zeta = K$. Therefore R/S is an $\mathfrak{X}$-group, and so

$$T \leqq S.$$

Hence η induces an epimorphism $\bar{\eta}\colon F/T \to H$ such that

$$\bar{\eta}\zeta = \bar{\theta}.$$

Moreover, $\bar{\eta}$ maps R/T to K. Hence we have a commutative diagram

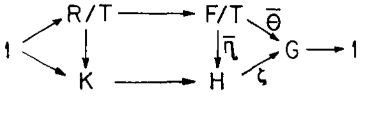

Fig. 3

as required.

As particular choices for $\mathfrak{X}$ in Theorem 3 we may take

(i) for any positive integers m and s, $\mathfrak{X}$ = the class of finite soluble groups of exponents dividing m and derived lengths $\leqq s$;

(ii) for any prime p, $\mathfrak{X}$ = the class of finite groups of exponent p (and 1): this by a famous theorem of A. I. Kostrikin [6].

Gaschütz's original result ([1, Satz 1]) corresponds to choosing $m = p$, a prime, and $s = 1$ in (i). Now in order to prove Theorem 1 we shall show that another possible choice for $\mathfrak{X}$ in Theorem 3 is

(iii) for any finite group K, $\mathfrak{X}$ = the smallest $\{\mathrm{S}, \mathrm{D}_0\}$-closed class of groups containing K.

Since a class of groups which is $\{\mathrm{S}, \mathrm{D}_0\}$-closed is certainly R_0-closed, what we have to show is that the smallest $\{\mathrm{S}, \mathrm{D}_0\}$-closed class of groups containing any finite group K is a bounded class. This is the content of Lemma 2.

LEMMA 2. *Let K be a finite group. Then the class of all subgroups of finite direct products of copies of K is a bounded class of groups.*

This follows from Theorem 15.71 of H. Neumann's book [7]. A direct proof is included here.

PROOF. We show that for every positive integer t, every t-generator subgroup H of a finite direct product of copies of K has order $\leqq |K|^{|K|^t}$. Let $H \leqq K_1 \times \cdots \times K_n$, where n is a positive integer and each K_j is a copy of $K (j = 1, \cdots, n)$. We argue by induction on n. The assertion is trivial for $n = 1$, so we suppose that $n > 1$. By the induction hypothesis we may assume that H is not isomorphic to a subgroup of a direct product of $n - 1$ copies of K. Let F be a free group of rank t and let

$$1 \to R \to F \overset{\theta}{\to} H \to 1$$

be a presentation of H. For $j = 1, \cdots, n$ let π_j be the projection homomorphism of $K_1 \times \cdots \times K_n$ onto K which maps each element of $K_1 \times \cdots \times K_n$ onto its jth component; and let ι denote the inclusion map of H in $K_1 \times \cdots \times K_n$. Then $\theta\iota\pi_1, \cdots, \theta\iota\pi_n$ are homomorphisms of F into K. Now if $\theta\iota\pi_r = \theta\iota\pi_s$ with $1 \leqq r < s \leqq n$ then, since θ maps F onto H, every element of H would have its rth and sth components equal; but then H would be isomorphic to a subgroup of a direct product of $n - 1$ copies of K, contrary to assumption. Therefore $\theta\iota\pi_1, \cdots, \theta\iota\pi_n$ are distinct homomorphisms of F into K. But since each homomorphism of F into K is determined by its effect on a set of t free generators of F, there are just $|K|^t$ distinct homomorphisms of F into K. Hence $n \leqq |K|^t$ and so

$$|H| \leqq |K|^n \leqq |K|^{|K|^t}.$$

This completes the induction proof.

We use also

LEMMA 3. *Let $\mathfrak{X}$ be any bounded, $\{\mathrm{S}, \mathrm{D}_0\}$-closed class of finite groups. Let n be a positive integer, G a finite n-generator group and $\mathscr{C}$ the class of extensions defined in Theorem 3. Let $1 \to K^* \to H^* \to G \to 1$ be an extension in $\mathscr{C}$ which is universal for $\mathscr{C}$. Then H^* splits over K^* if and only if all $\mathfrak{X}$-groups have orders co-prime to $|G|$.*

PROOF. If $(|G|, |K^*|) = 1$ then H^* splits over K^*, by the Schur-Zassenhaus theorem.

Now suppose that there is an $\mathfrak{X}$-group J such that $(|G|, |J|) > 1$. Let p be a common prime divisor of $|G|$ and $|J|$. Since $\mathfrak{X}$ is $\{\mathrm{S}, \mathrm{D}_0\}$-closed, $\mathfrak{X}$ contains all finite elementary abelian p-groups. By a result of Gaschütz [1] mentioned above, there is an extension

$$1 \to A \to H \to G \to 1,$$

where A is an elementary abelian p-group and $1 < A \leqq \Phi(H)$. It follows from this, since G is n-generator, that H is n-generator, and therefore that the extension belongs to $\mathscr{C}$. Hence there is an epimorphism $\chi: H^* \to H$ making a commutative diagram.

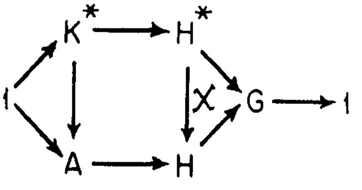

Fig. 4

Then, since $\mathrm{Ker}\,\chi \leqq K^*$, if H^* were to split over K^* it would follow that H split over A, which is false. Hence H^* does not split over K^*

PROOF OF THEOREM 1. We suppose that G and K are finite groups such that $(|G|, |K|) > 1$. Let $\mathfrak{X}$ be the class of all subgroups of finite direct products of copies of K; thus $\mathfrak{X}$ is the smallest $\{\mathrm{S}, \mathrm{D}_0\}$-closed class of groups containing K. By Lemma 2, $\mathfrak{X}$ is a bounded class. Since also $\mathfrak{X}$ is R_0-closed, Theorem 3 is applicable. Let n be a positive integer such that the direct product $G \times K$ is an n-generator group and let $\mathscr{C}$ be the class of extensions defined in Theorem 3, with $\mathfrak{X}$ as above. Let

$$1 \to K^* \to H^* \to G \to 1$$

be an extension in $\mathscr{C}$ which is universal for $\mathscr{C}$. There is also in $\mathscr{C}$ an extension

$$1 \to K \to G \times K \to G \to 1 .$$

Hence there is a commutative diagram

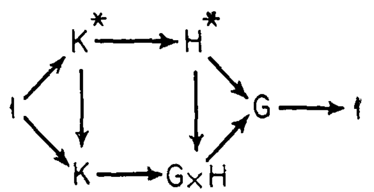

Fig. 5

in which K^* is mapped onto K. Thus K is an epimorphic image of K^*, which is a subgroup of a finite direct product of copies of K. Moreover,

$$1 \to K^* \to H^* \to G \to 1$$

is an extension of K^* by G which, by Lemma 3, does not split.

To prove Theorem 2, we note first

LEMMA 4. *Let n be a positive integer and let $L_1, \cdots, L_n$ be normal subgroups of, respectively, groups $K_1, \cdots, K_n$. Then the direct product $K_1 \times \cdots \times K_n$ splits over $L_1 \times \cdots \times L_n$ if and only if each K_j splits over L_j, for $j = 1, \cdots, n$.*

PROOF. Let $K = K_1 \times \cdots \times K_n$ and $L = L_1 \times \cdots \times L_n$. If J_j is a complement to L_j in K_j for $j = 1, \cdots, n$ then clearly $J_1 \times \cdots \times J_n$ is a complement to L in K. Conversely, if J is a complement to L in K then, for $j = 1, \cdots, n$, $(JL^j) \cap K_j$ is a complement to L_j in K_j, where L^j is the product of all the L_i's except L_j.

LEMMA 5. Let *K be a group with a normal subgroup L, and let n be a positive integer. Let W denote the natural wreath product of K by Σ_n, the symmetric group of degree n, let $K_1 \times \cdots \times K_n$ denote the base group of W (a direct product of n copies of K) and let $L_1 \times \cdots \times L_n$ denote the corresponding direct product of n copies of L, which is a normal subgroup of W. Then W splits over $L_1 \times \cdots \times L_n$ if and only if K splits over L.*

PROOF. If W splits over $L_1 \times \cdots \times L_n$ then $K_1 \times \cdots \times K_n$ splits over $L_1 \times \cdots \times L_n$, and so, by Lemma 4, K splits over L.

Conversely, suppose that K splits over L, and let J be a complement to L in K. Let $J_1 \times \cdots \times J_n$ denote the corresponding direct product of n copies of J which is a subgroup of W normalized by Σ_n. Now it is clear that $(J_1 \times \cdots \times J_n)\Sigma_n$ is a subgroup of W which is a complement to $L_1 \times \cdots \times L_n$ in W.

Now let $\mathfrak{Y}$ denote the class of finite groups all extensions of which split. A finite group K is a $\mathfrak{Y}$-group if and only if $Z(K) = 1$ and Aut K splits over Inn K: see [8, Corollary 2.3].

LEMMA 6. *Let K be any non-trivial $\mathfrak{Y}$-group. Then any indecomposable direct factor of K is also a $\mathfrak{Y}$-group.*

PROOF. Say $K = K_{11} \times \cdots \times K_{1r_1} \times K_{21} \times \cdots \times K_{2r_2} \times \cdots \times K_{s1} \times \cdots \times K_{sr_s}$ where $s, r_1, r_2, \cdots, r_s$ are positive integers, each K_{ij} is a directly indecomposable group and $K_{ij} \cong K_{i'j'}$ if and only if $i = i'$, for $1 \leqq i,\ i' \leqq s,\ 1 \leqq j \leqq r_i$, $1 \leqq j' \leqq r_{i'}$. Since by hypothesis $Z(K) = 1$, it follows that $Z(K_{ij}) = 1$ for all i,j. Also, by the Krull-Remak-Schmidt theorem ([5, I.12.6]) the decomposition of K above is the unique decomposition of K as a direct product of indecomposable factors. Hence, for $i = 1, \cdots, s$, $K_{i1} \times \cdots \times K_{ir_i}$ is a characteristic subgroup of K; and

$$\operatorname{Aut} K \cong W_1 \times W_2 \times \cdots \times W_s,$$

where, for $i = 1, \cdots, s$, W_i is the natural wreath product of $\operatorname{Aut} K_{i1}$ by Σ_{r_i}. The normal subgroup of $W_1 \times \cdots \times W_s$ corresponding to $\operatorname{Inn} K$ is $Y_1 \times \cdots \times Y_s$, where, for $i = 1, \cdots, s$, Y_i is the direct product of r_i copies of $\operatorname{Inn} K_{i1}$ naturally contained in the base group of W_i. By hypothesis, Aut K splits over Inn K. Hence, by Lemma 4, W_i splits over Y_i, for $i = 1, \cdots, s$; then, also by Lemma 5, $\operatorname{Aut} K_{i1}$ splits over $\operatorname{Inn} K_{i1}$. Hence, for $i = 1, \cdots, s$, K_{i1} is a $\mathfrak{Y}$-group. This proves the lemma.

PROOF OF THEOREM 2. We have to show that if K_1 and K_2 are $\mathfrak{Y}$-groups then the direct product $K_1 \times K_2$ is a $\mathfrak{Y}$-group. Each of K_1 and K_2 can be expressed (by the Krull-Remak-Schmidt theorem uniquely) as a direct product of indecomposable factors; and by Lemma 6, these indecomposable factors are also $\mathfrak{Y}$-groups. Hence, in order to prove Theorem 2, it is enough to show that any finite direct product of directly indecomposable $\mathfrak{Y}$-groups is a $\mathfrak{Y}$-group.

Let $K = K_{11} \times \cdots \times K_{1r_1} \times K_{21} \times \cdots \times K_{2r_2} \times \cdots \times K_{s1} \times \cdots \times K_{sr_s}$, where $s, r_1, r_2, \cdots, r_s$ are positive integers, each K_{ij} is a directly indecomposable $\mathfrak{Y}$-group and $K_{ij} \cong K_{i'j'}$ if and only if $i = i'$, for $1 \leqq i, i' \leqq s$, $1 \leqq j \leqq r_i$, $1 \leqq j' \leqq r_{i'}$. Since $Z(K_{ij}) = 1$ for all i, j, $Z(K) = 1$. Also, as in the proof of Lemma 6,

$$\operatorname{Aut} K \cong W_1 \times W_2 \times \cdots \times W_s,$$

where for $i = 1, \cdots, s$, W_i is the natural wreath product of $\operatorname{Aut} K_{i1}$ by Σ_{r_i}. As before, let the subgroup of $W_1 \times \cdots \times W_s$ corresponding to $\operatorname{Inn} K$ be $Y_1 \times \cdots \times Y_s$, where Y_i is the direct product of r_i copies of $\operatorname{Inn} K_{i1}$. Since $\operatorname{Aut} K_{i1}$ splits over $\operatorname{Inn} K_{i1}$, Lemma 5 shows that W_i splits over Y_i. Then by Lemma 4, $W_1 \times \cdots \times W$ splits over $Y_1 \times \cdots \times Y_s$. Hence K is a $\mathfrak{Y}$-group, as required.

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